Math 241 - Exam 3

Tuesday, Nov 7th, 2017

You have 50 minutes to complete this exam. Do not simplify unless indicated. Calculators are **not** allowed. Submit each of the five problems on a separate sheet. Please cross out any work you don't want to be graded.

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(1) (a) **[10pts]** Perform a change of coordinantes to polar and evaluate the following integral

$$\int_0^{\sqrt{3}} \int_{\frac{1}{\sqrt{3}}x}^{\sqrt{4-x^2}} e^{x^2+y^2} \mathrm{d}y \mathrm{d}x.$$

Solution: The region integrated over is the region of a disk with angles from $\theta = \pi/3$ to $\theta = \pi/2$. This can be easily changed to polar coordinates

$$\int_{\pi/3}^{\pi/2} \int_0^2 re^{r^2} \mathrm{d}r \mathrm{d}\theta = \frac{\pi}{12}(e^4 - 1).$$

(b) **[10pts]** Evaluate the following iterated integral

$$\int_0^1 \int_x^1 e^{y^2} \mathrm{d}y \mathrm{d}x.$$

Solution: To evaluate this integral, we will find it easier to change the order of integration. The region is written as a vertically simple one as $x \leq y \leq 1$, $0 \leq x \leq 1$. However it can also be written as horizontally simple $0 \leq x \leq y$, $0 \leq y \leq 1$. The integral becomes

$$\int_0^1 \int_0^y e^{y^2} \mathrm{d}x \mathrm{d}y = \int_0^1 ye^{y^2} \mathrm{d}y = \frac{1}{2}(e^1 - 1)$$

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(2) **[20 pts]** Consider the (ice cream cone) region D inside the sphere $x^2+y^2+(z-1)^2 = 1$ and above the cone $z = \sqrt{3x^2+3y^2}$. Find the volume of the region D by evaluating an iterated integral in spherical coordinates. *Hint: The sphere above can be written in spherical coordinates as $\rho = f(\phi)$. What is f?*

Solution: The sphere can be written as $\rho = 2\cos\phi$ and the cone $\phi = \pi/6$. Volume can be written as an iterated integral in spherical coordinates.

$$\begin{aligned} V = \iiint_D 1 \mathrm{d}V &= \int_0^{2\pi} \int_0^{\pi/6} \int_0^{2\cos\phi} \rho^2 \sin\phi \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta \\ &= \frac{16\pi}{3} \int_0^{\pi/6} (\cos\phi)^3 \sin\phi \, \mathrm{d}\phi \\ &= \frac{4\pi}{3}(-(\cos\phi)^4|_0^{\pi/6}) = \frac{4\pi}{3}(1 - \frac{9}{16}) = \frac{7\pi}{12} \end{aligned}$$

(3) (a) [**10 pts**] Let D be the region inside the cylinder $(x-2)^2+y^2=4$ and between the two planes $z=2$, $2x-y+z=8$. Write the following integral $\iiint_D xy\mathrm{d}V$ as an iterated triple integral in cylindrical coordinates. **Do not evaluate**

Solution: The region D in cylindrical coordinates is given by $2 \leq z \leq 8 - 2r\cos\theta + r\sin\theta$, $0 \leq r \leq 4\cos\theta$, $-\pi/2 \leq \theta \leq \pi/2$. Therefore the integral can be written as

$$\iiint_D xy\,\mathrm{d}V = \int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta}\int_2^{8-2r\cos\theta+r\sin\theta} r^3\cos\theta\sin\theta\mathrm{d}z\mathrm{d}r\mathrm{d}\theta.$$

(b) [**10 pts**] Let D be the region in the first octant between the two spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ and below $z=\sqrt{x^2+y^2}$. Write the volume of the region as an iterated triple integral in spherical coordinates. **Do not evaluate**

Solution: The region D can be described by $1 \leq \rho \leq 2$, $\pi/4 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$. Therefore the volume is given by

$$\iiint_D 1\,\mathrm{d}V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_1^2 \rho^2\sin\phi\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta.$$

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(4) In the following problems, give a smooth parameterization $\mathbf{r}(u,v)$ of the following surfaces (be sure describe the region that the parameters (u,v) take values in).

(a) [**10 pts**] The surface of revolution obtained by revolving the function $f(z)$ about the z axis for $0 \leq z \leq 2$.

Solution: The surface of revolution is given by

$$\mathbf{r}(\theta,z) = f(z)\cos\theta\mathbf{i} + f(z)\sin\theta\mathbf{j} + z\mathbf{k}.$$

(a) [**10 pts**] The torus $\rho = \sin\phi$ (written here in spherical coordinates). Recall x, y, z described in terms of spherical coordinates

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

Since $\rho = \sin\phi$ defines a relation between ρ and ϕ, we can substitute this into the expressions above to obtain a parameterization in the θ ϕ variables,

$$\mathbf{r}(\theta,\phi) = \sin\phi^2\cos\theta\mathbf{i} + \sin\phi^2\sin\theta\mathbf{j} + \sin\phi\cos\phi\mathbf{k}.$$

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(5) [**20 pts**] Let R be the region bounded by the ellipse $\frac{x^2}{9}+y^2=4$. Perform a change of variables to transform the integral

$$\iint_R e^{x+y}\,\mathrm{d}A$$

to an integral over the region S bounded by the circle $x^2 + y^2 = 1$. Write the integral in polar coordinates but **do not evaluate**.

Solution: To change variables, write the ellipse as

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{2}\right)^2 = 1.$$

We then choose $u = x/6$ and $v = y/2$, so that $x = 6u$ and $y = 2v$. The Jacobian is therefore

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12.$$

Therefore we can change the integral to

$$\iint_R e^{x+y}\,\mathrm{d}A = 12 \iint_S e^{6u+2v}\,\mathrm{d}A,$$

where S is the region inside of the circle $u^2+v^2 = 1$. In polar coordinates, this becomes

$$12 \int_0^{2\pi} \int_0^1 e^{6r\cos\theta + 2r\sin\theta} r\mathrm{d}r\mathrm{d}\theta.$$

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(6) **Bonus [10pts]** Consider the region D inside the sphere $x^2 + y^2 + z^2 = 4$ and above the plane $z = 1$. Write the integral $\iiint_D (x^2 + y^2 + z^2)^{-1}\mathrm{d}V$ as an iterated integral in spherical coordinates and evaluate.

Solution: The region in spherical coordinates is given by $\sec\phi \leq \rho \leq 2$, $0 \leq \phi \leq \pi/3$, $0 \leq \theta \leq 2\pi$. The reason that ϕ ranges between 0 an $\pi/3$ is because the sphere $\rho = 2$ and the plane $\rho\cos\phi = 1$ intersect when $\cos\phi = 1/2$ (i.e $\phi = \pi/3$). The integral can be written in spherical coordinates as

$$\begin{aligned}
\iiint_D \rho^{-2}\mathrm{d}V &= \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^2 \sin\phi\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta \\
&= 2\pi \int_0^{\pi/3} (2\sin\phi - \tan\phi)\,\mathrm{d}\phi \\
&= 2\pi(-2\cos\phi|_0^{\pi/3} + \ln(\cos\phi)|_0^{\pi/3}) \\
&= 2\pi(1 - \log 2).
\end{aligned}$$