Math 241 - Exam 4

Monday, Dec 4th, 2017

You have 50 minutes to complete this exam. Do not simplify unless indicated. Calculators are **not** allowed. Submit each of the five problems on a separate sheet. Please cross out any work you don't want to be graded.

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(1) (a) [**10 pts**] Evaluate the integral $\int_C 3y^2 \,\mathrm{d}s$ where C is the top half of a circle of radius 2 in the x, y plane.

Solution: The top half of the circle can be parameterized by $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$, $0 \le t \le \pi$, which satisfies $\|\mathbf{r}'(t)\| = 2$. The integral is then

$$\int_C 3y^2 \,\mathrm{d}s = 24\int_0^\pi (\cos t)^2 \mathrm{d}t = 12\pi.$$

(b) [**10 pts**] Evaluate the line integral

$$\int_C (2xy + z)\mathrm{d}x + (x^2 + 2zy)\mathrm{d}y + (y^2 + x + 1)\mathrm{d}z,$$

where C is parameterized by $\mathbf{r}(t) = \sin\left(\frac{\pi}{2}t\right)\mathbf{i} + 2te^{1-t}\mathbf{j} + \sqrt{1-t}\,\mathbf{k}, \quad 0 \le t \le 1.$

Solution: Note that the vector field is conservative since

$$M_y = 2x = N_x, \quad M_z = 1 = P_x, \quad N_z = 2y = P_y.$$

The potential function can easily be seen to be given by

$$f(x, y, z) = x^2y + y^2z + xz + z.$$

The start and end points of the curve are given by

$$P = \mathbf{r}(0) = (0, 0, 1), \quad Q = \mathbf{r}(1) = (1, 2, 0).$$

By the fundamental theorem of line integrals, the line integral is simply given by

$$f(1, 2, 0) - f(0, 0, 1) = 2 - 1 = 1.$$

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(2) [**20 pts**] Let Σ be the part of the paraboloid $z = y^2 - x^2$ that intersects the cylinder $r = 2\sin\theta$. Parameterize the surface Σ and write the following surface integral

$$\iint_\Sigma x^2 z \,\mathrm{d}S$$

as an iterated double integral. *DO NOT EVALUATE.*

Solution: The surface can be parameterized by

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (y^2 - x^2)\mathbf{k}.$$

for x, y inside the region R with boundary $r = 2\sin\theta$. Since this is the graph of a function we can easily compute

$$\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{4x^2 + 4y^2 + 1}.$$

The surface intgral can then be written as

$$\iint_{\Sigma} x^2 z \, \mathrm{d}S = \iint_{R} x^2(y^2 - x^2)\sqrt{4x^2 + 4y^2 + 1} \, \mathrm{d}A.$$

Using Polar coordinates, this integral can be written as an iterated integral

$$\int_0^{\pi} \int_0^{2\sin\theta} r^5 (\cos\theta)^2 [(\sin\theta)^2 - (\cos\theta)^2] \sqrt{4r^2 + 1} \, \mathrm{d}r \mathrm{d}\theta.$$

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(3) **[20 pts]** Evaluate the line integral $\int_C 4xy \, \mathrm{d}x + x^2 \, \mathrm{d}y$, where C is the closed counter-clockwise oriented curve in the x, y plane defined as the boundary of the part of the disk of radius 2 that lies in the first quadrant.

Solution: Since the curve is closed, we use Green's Theorem

$$\int_C 4xy \, \mathrm{d}x + x^2 \, \mathrm{d}y = -\iint_R 2x \, \mathrm{d}A,$$

where R is the part of the disk of radius 2 in the first quadrant. Writing this in polar coordinates allows us to evaluate it.

$$-2 \int_0^{\pi/2} \int_0^2 r^2 \cos\theta \mathrm{d}r \mathrm{d}\theta = -2 \left(\int_0^2 r^2 \, \mathrm{d}r \right) \left(\int_0^{\pi/2} \cos\theta \, \mathrm{d}\theta \right) = -\frac{16}{3}$$

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(4) **[20 pts]** Consider the curve C given by the triangle with vertices $(1, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 3)$ oriented in the counter-clockwise direction when viewed from above. Apply Stokes' Theorem to the line integral

$$\int_C (xz\mathbf{i} - yz\mathbf{j} + xyz\mathbf{k}) \cdot \mathrm{d}\mathbf{r},$$

and write the resulting flux integral as a double iterated integral. *DO NOT EVALUATE.*

Solution: Let Σ be the part of the plane that lies inside the trianle formed by the three points, and let it have upward facing normal. The plane can be described by $3x + y + z = 3$, where x and y are restricted to lie beneath the line $3x + y = 3$ and in the first quadrant. To apply Stokes theorem, we take the curl of the vector field $\mathbf{F} = xz\mathbf{i} - yz\mathbf{j} + xyz\mathbf{k}$,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -yz & xyz \end{vmatrix} = (xz + y)\mathbf{i} + (x - yz)\mathbf{j}.$$

Therefore, Stokes theorem implies

$$\int_C (xz\mathbf{i} - yz\mathbf{j} + xyz\mathbf{k}) \cdot \mathrm{d}\mathbf{r} = \iint_\Sigma ((xz + y)\mathbf{i} + (x - yz)\mathbf{j}) \cdot \mathbf{n}\mathrm{d}S$$

$$= \int_0^1 \int_0^{3-3x} [(x(3 - y - 3x) + y)\mathbf{i} + (x - y(3 - 3x - y))\mathbf{j}] \cdot (3\mathbf{i} + \mathbf{j} + \mathbf{k})\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 \int_0^{3-3x} (10x - 9x^2 + y^2)\, \mathrm{d}y\mathrm{d}x.$$

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(5) **[20 pts]** Let Σ be the closed surface comprised of the piece of the cone $z = \sqrt{x^2 + y^2}$ that lies between the two spheres of radius 1 and radius 3 as well as the parts of those two sphere that lie above the cone. Assume that Σ is oriented with outward facing normal $\mathbf{n}$. Apply the Divergence Theorem to the flux integral

$$\iint_\Sigma (x^2y\mathbf{i} + xy^2\mathbf{j} + xyz\mathbf{k}) \cdot \mathbf{n}\, \mathrm{d}S,$$

and write the resulting integral as a triple iterated integral. *DO NOT EVALUATE.*

Solution: Define D to be the region inside of the closed surface Σ. This can be described in spherical coordinates by $1 \leq \rho \leq 3$, $0 \leq \phi \leq \pi/4$, $0 \leq \theta \leq 2\pi$. Using the divergence theorem, we obtain

$$\iint_\Sigma (x^2y\mathbf{i} + xy^2\mathbf{j} + xyz\mathbf{k}) \cdot \mathbf{n}\, \mathrm{d}S = \iiint_D 5xy\, \mathrm{d}V$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_1^3 5\rho^4(\sin\phi)^3 \cos\theta \sin\theta \mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta.$$