MATH 241 Final Examination

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Instructions. Answer each question **on a *separate* answer sheet**. Show all your work. A correct answer without work to justify it may not receive full credit. *Be sure your name, section number, and problem number are on each answer sheet, and that you have copied and signed the honor pledge on the first answer sheet.* The point value of each problem is indicated. The exam is worth a total of 200 points. In problems with multiple parts, whether the parts are related or not, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. Please leave answers such as $5\sqrt{2}$ or 3π in terms of radicals and π and *do not convert to decimals.*
You are allowed use of one sheet of notes. Calculators are not permitted.

1. (20 points, divided as indicated) Consider the vectors $\mathbf{a} = 2\mathbf{i}+\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = \mathbf{i}+\mathbf{j}-\mathbf{k}$, and $\mathbf{c} = \mathbf{i}-\mathbf{j}+9\mathbf{k}$.
   (a) (5 points) Show that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.
   (b) (5 points) Given that $\mathbf{c}$ lies in the same plane as $\mathbf{a}$ and $\mathbf{b}$, resolve $\mathbf{c}$ into two vectors, one parallel to $\mathbf{a}$ and one parallel to $\mathbf{b}$.
   (c) (10 points) Find the equation of the plane that passes through the origin and contains $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

2. (25 points) Find the critical points of the function $f(x,y) = x^4 + 2x^2y - y^2 - 4y$ and classify each one as a local maximum point, local minimum point, saddle point, or something else.

3. (30 points) Let D be the solid region above the x-y plane, inside the cone $z = 9 - \sqrt{x^2+y^2}$, and inside the cylinder $r = \sin\theta$. (Here r and θ are the usual cylindrical coordinates.) Find the volume of D.

4. (20 points, divided as indicated) Consider the cycloid C parametrized by

$$\mathbf{r}(t) = (t - \sin(t))\mathbf{i} + (1 - \cos(t))\mathbf{j} \text{ for } -5\pi \leq t \leq 5\pi.$$

   (a) (10 points) Is C smooth, piecewise smooth, or neither? Explain.
   (b) (10 points) Compute the unit tangent vector $\mathbf{T}(t)$ at the point $\mathbf{r}(t)$. What are the limits of $\mathbf{T}(t)$ as t approaches 0 from the right and from the left, and how does this relate to your answer to part (a)?

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5. (25 points, divided as indicated) Let $f(x, y, z) = xyz + z^3$ and let S be the surface given by the equation $f(x, y, z) = 12$.

   (a) (15 points) Find the tangent plane to the surface S at the point $(-2, -1, 2)$.

   (b) (10 points) Compute the directional derivative $D_{\mathbf{u}}f(-2, -1, 2)$ if $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

6. (25 points) Let R be the region in the first quadrant bounded by the curves $y = \frac{1}{2}x$, $y = 3x$, $y = \frac{1}{x}$, and $y = \frac{4}{x}$. Use the change of variables $u = \frac{y}{x}$ and $v = xy$ to evaluate the integral $\iint_R y \; dA$. Proceed until you have an iterated integral in u and v with explicit limits of integration, but do not evaluate.

7. (25 points) Evaluate the surface integral $\iint_\Sigma \mathbf{F} \cdot \mathbf{n} \, dS$, where

$$\mathbf{F}(x, y, z) = -x \sec^2 y \, \mathbf{i} + \tan y \, \mathbf{j} + z \, \mathbf{k},$$

   Σ is the surface consisting of the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and the disk $x^2 + y^2 \leq 9$ in the x-y plane, and $\mathbf{n}$ is the outward-pointing unit normal vector.

8. (30 points) Evaluate the line integral

$$\int_C [e^{(x^2)} - 4xy] \, dx + 2x \, dy,$$

   where C is the triangle that lies in the x-y plane, has vertices $(0, 0)$, $(2, 0)$ and $(0, 2)$, and is oriented counterclockwise.