Math 241 Fall 2014 Final Exam Solutions

1. Given the line $\mathcal{L}$ with symmetric equation $x = \frac{y-1}{3} = \frac{z}{2}$, the plane with equation $\mathcal{P}$ given by $9x - 2y - z = 0$ and the point $\mathcal{Q} = (1, -2, 5)$:

   (a) Determine whether the line $\mathcal{L}$ is parallel to the plane $\mathcal{P}$. [10 pts]

   Solution:

   The direction vector for $\mathcal{L}$ is $\bar{L} = 1\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$.

   The normal vector for $\mathcal{P}$ is $\bar{N} = 9\,\mathbf{i} - 2\,\mathbf{j} - 1\,\mathbf{k}$.

   We check $\bar{L} \cdot \bar{N} = (1)(9) + (3)(-2) + (2)(-1) = 1 \neq 0$ so the line is not parallel to the plane.

   (b) Find the distance from the point $\mathcal{Q}$ to the line $\mathcal{L}$. Simplify. [15 pts]

   Solution:

   Pick a point on the line: $\mathcal{R} = (0, 1, 0)$.

   Then

$$\overline{\mathcal{RQ}} = 1\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k}$$

   and so

$$\text{dist} = \frac{||\overline{\mathcal{RQ}} \times \bar{L}||}{||\bar{L}||} = \frac{||-21\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}||}{||1\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}||} = \frac{\sqrt{(-21)^2 + 3^2 + 6^2}}{\sqrt{1^2 + 3^2 + 2^2}}$$

2. Let the position of an object in motion be given by $\bar{r}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j} + e^t\,\mathbf{k}$.

   (a) Find the velocity and acceleration of the object at any t. [10 pts]

   Solution:

   We calculate

$$\bar{r}'(t) = (e^t \cos t - e^t \sin t)\,\mathbf{i} + (e^t \sin t + e^t \cos t)\,\mathbf{j} + e^t\,\mathbf{k}$$

   and

$$\begin{aligned}\bar{r}''(t) &= (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\,\mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\,\mathbf{j} + e^t k \\ &= -2e^t \sin t\,\mathbf{i} + 2e^t \cos t\,\mathbf{j} + e^t\,\mathbf{k}\end{aligned}$$

   (b) Write down the integral for the distance traveled by the object between $t = 0$ and $t = 2$ but do not evaluate. [5 pts]

   Solution:

   We have

$$||r'(t)|| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2}$$

   so that the length is

$$\int_0^2 \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2}\ dt$$

   (c) Compute the curvature of the object's path at $t = 0$. [10 pts]

   Solution:

   We have

$$\bar{r}'(0) = 1\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k}$$

   and

$$\bar{r}''(0) = 0\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}$$

   so that

$$\kappa = \frac{||(1\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k}) \times (0\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k})||}{||1\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k}||^3} = \frac{||-1\,\mathbf{i} - 1\,\mathbf{j} + 2\,\mathbf{k}||}{||1\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k}||^3} = \frac{\sqrt{6}}{3^{3/2}}$$

3. Use the method of Lagrange Multipliers to determine the maximum and minimum values of the function $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 4$. You may assume that the maximum and minimum exist. [25 pts]

Solution:

We assign $g(x, y) = 4x^2 + y^2$ and then we set up the system:

$$y = \lambda(8x) \tag{1}$$
$$x = \lambda(2y) \tag{2}$$
$$4x^2 + y^2 = 4 \tag{3}$$

Equation (1) tells us that $\lambda = \frac{y}{8x}$ unless $x = 0$ (but $x = 0$ would give us $y = 0$ in (1) and this contradicts (3)).

Equation (2) tells us that $\lambda = \frac{x}{2y}$ unless $y = 0$ (but $y = 0$ would give us $x = 0$ in (1) and this contradicts (3)).

Thus $\frac{y}{8x} = \frac{x}{2y}$ and so $2y^2 = 8x^2$ or $y^2 = 4x^2$.

Plugging this into (3) yields $8x^2 = 4$ so $x = \pm\sqrt{1/2}$.

If $x = \sqrt{1/2}$ then (3) tells us $x = \pm\sqrt{2}$ and the same for the other x.

Thus we have four points: $(-\sqrt{1/2}, -\sqrt{2})$, $(-\sqrt{1/2}, +\sqrt{2})$, $(+\sqrt{1/2}, -\sqrt{2})$, and $(+\sqrt{1/2}, +\sqrt{2})$

Next:

$$f(-\sqrt{1/2}, -\sqrt{2}) = 1$$
$$f(-\sqrt{1/2}, +\sqrt{2}) = -1$$
$$f(+\sqrt{1/2}, -\sqrt{2}) = -1$$
$$f(+\sqrt{1/2}, +\sqrt{2}) = 1$$

Thus the minimum is -1 and the maximum is 1.

4. (a) Let $D(x, y) = 300 - 2x^2 - 3y^2$ denote the depth of a lake in feet. If a boat is at $(3, 5)$, in what direction should the boat travel for the depth of the water to increase most rapidly and what would that rate of increase be? [10 pts]

Solution:

We have

$$\nabla D(x, y) = -4x\,\mathbf{i} - 6y\,\mathbf{j}$$

and so the direction would be

$$\nabla D(3, 5) = -12\,\mathbf{i} - 30\,\mathbf{j}$$

and the rate of increase would be

$$||\nabla D(3, 5)|| = \sqrt{(-12)^2 + (-30)^2}$$

(b) Ohm's Law states that $I = \frac{V}{R}$ which relates current (I) with voltage (V) and resistance (R). Suppose the voltage is decreasing at 5 volts/second while the resistance is decreasing at 2 ohms/second. Find the rate of change of the current with respect to time when the voltage is 80 volts and the resistance is 40 ohms. [15 pts]

Solution:

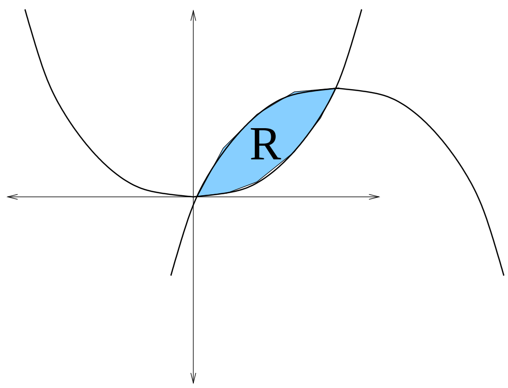
The chain rule tells us that

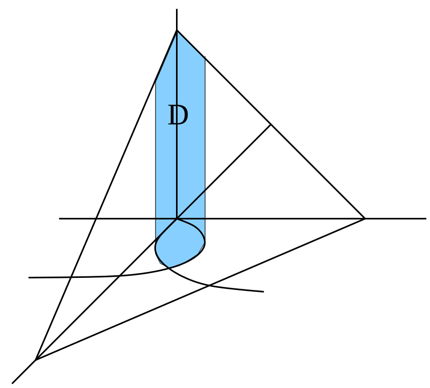
$$\begin{aligned}
\frac{dI}{dt} &= \frac{\partial I}{\partial V}\frac{\partial V}{\partial t} + \frac{\partial I}{\partial R}\frac{\partial R}{\partial t} \\
&= \frac{1}{R}(-5) - \frac{V}{R^2}(-2) \\
&= \frac{1}{40}(-5) - \frac{80}{40^2}(-2)
\end{aligned}$$

5. Let R be the region in the xy-plane above the graph of $y = x^2$ and below the graph of $y = -(x-1)^2 + 1$. Let D be the solid above R and below the plane $x + y + z = 5$.

(a) Separately sketch reasonable pictures of both R and D. [10 pts]

Solution:





(b) Set up an iterated double integral for the volume of D. Do not evaluate. [15 pts]

Solution:

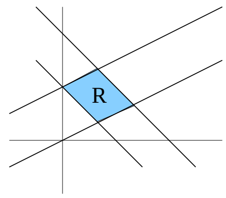
If we parametrize R as vertically simple then we get

$$\int_0^1 \int_{x^2}^{-(x-1)^2+1} 5 - x - y \; dy \; dx$$

6. Let R be the parallelogram in the xy-plane formed by the lines $x + y = 1$, $x + y = 2$, $2y - x = 2$ and $2y - x = 0$.

   (a) Sketch R. [5 pts]

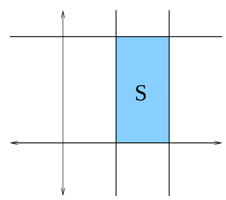
   Solution:



   (b) Use a change of variables to evaluate $\iint_R x + y \; dA$. Make sure to draw the new region in the uv-plane. This integral must be evaluated! [20 pts]

   Solution:

   We substitute $u = x + y$ and $v = 2y - x = -x + 2y$. This gives us the new region S bounded by the lines $u = 1, 2$ and $v = 0, 2$:



   Then we solve to get $x = \frac{2}{3}u - \frac{1}{3}v$ and $y = \frac{1}{3}u + \frac{1}{3}v$ so that

$$J = \begin{vmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{vmatrix} = 1/3$$

   Alternately without solving for x and y:

$$J = 1 \div \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 1/3$$

   Then

$$\begin{aligned} \iint_R x + y \; dA &= \iint_S u|1/3| \; dA \\ &= \int_1^2 \int_0^2 \frac{1}{3}u \; dv \; du \\ &= \int_1^2 \frac{1}{3}uv\Big|_0^2 \; du \\ &= \int_1^2 \frac{2}{3}u \; du = \frac{1}{3}u^2\Big|_1^2 = \frac{1}{3}(4) - \frac{1}{3}(1) \end{aligned}$$

7. Let C be the edge of the part of the plane $2x + 2y + z = 10$ in the first octant, oriented counterclockwise when viewed from above.

(a) Apply Stokes' Theorem to the integral $\int_C 2y\ dx + x\ dy + xz\ dz$ to get a surface integral over a surface Σ. Describe Σ, including its induced orientation. Either words or a picture suffice. [5 pts]

Solution:

We have

$$\int_C 2y\ dx + x\ dy + xz\ dz = \iint_\Sigma [(0-0)\,\mathbf{i} - (z-0)\,\mathbf{j} + (1-2)\,\mathbf{k}] \cdot \bar{n}\ dS$$

where Σ is the portion of the plane $2x + 2y + z = 10$ in the first octant, oriented up and out.

(b) Parametrize Σ and convert your answer to (a) to an iterated double integral. [15 pts]

Solution:

Parametrize Σ as: $\bar{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (10 - 2x - 2y)\,\mathbf{k}$ where $0 \le x \le 5$ and $0 \le y \le 5 - x$. Then

$$\begin{aligned} \bar{r}_x &= 1\,\mathbf{i} + 0\,\mathbf{j} - 2\,\mathbf{k} \\ \bar{r}_y &= 0\,\mathbf{i} + 1\,\mathbf{j} - 2\,\mathbf{k} \\ \bar{r}_x \times \bar{r}_y &= 2\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k} \end{aligned}$$

these vectors match the orientation for Σ and so

$$\begin{aligned} &\iint_\Sigma [(0-0)\,\mathbf{i} - (z-0)\,\mathbf{j} + (1-2)\,\mathbf{k}] \cdot \bar{n}\ dS \\ &= +\iint_R [0\,\mathbf{i} - (10 - 2x - 2y)\,\mathbf{j} - 1\,\mathbf{k}] \cdot [2\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}]\ dA \\ &= \int_0^5 \int_0^{5-x} 4x + 4y - 21\ dy\ dy \end{aligned}$$

(c) Evaluate. [5 pts]

Solution:

$$\begin{aligned} \int_0^5 \int_0^{5-x} 4x + 4y - 21\ dy\ dx &= \int_0^5 4xy + 2y^2 - 21y \Big|_0^{5-x}\ dx \\ &= \int_0^5 4x(5-x) + 2(5-x)^2 - 21(5-x)\ dx \\ &= \int_0^5 20x - 4x^2 + 2x^2 - 20x + 50 - 105 + 21x\ dx \\ &= \int_0^5 -2x^2 + 21x - 55\ dx \\ &= -\frac{2}{3}x^3 + \frac{21}{2}x^2 - 55x \Big|_0^5 = -\frac{2}{3}(5)^3 + \frac{21}{2}(5)^2 - 55(5) \end{aligned}$$

8. (a) Let C be the part of the graph of the function $y = x^2$ from $x = 1$ to $x = 2$. Write down the iterated single integral corresponding to $\int_C x - y\ ds$. Do not evaluate. [10 pts]

Solution:

We parametrize as $\bar{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$ for $1 \le t \le 2$.

Then $\bar{r}'(t) = 1\,\mathbf{i} + 2t\,\mathbf{j}$ and $||\bar{r}'(t) = \sqrt{5}||$.

Thus

$$\int_C x - y\ ds = \int_1^2 (t - t^2)\sqrt{5}\ dt$$

(b) Let D be the solid inside the cone with spherical equation $\phi = \frac{\pi}{6}$ and below the plane $z = 3$. Let Σ be the surface of D oriented inwards. Apply the Divergence Theorem to the surface integral $\iint_\Sigma (x\,\mathbf{i} + xz\,\mathbf{j} + z^2\,\mathbf{k}) \cdot \bar{n}\ dS$ and then use a spherical parametrization to obtain a triple iterated integral. Do not evaluate. [15 pts]

Solution:

By the Divergence Theorem (and due to orientation)

$$\iint_\Sigma (x\,\mathbf{i} + xz\,\mathbf{j} + z^2\,\mathbf{k}) \cdot \bar{n}\ dS = -\iiint_D 1 + 0 + 2z\ dV$$

where V is the solid inside the cone and below the plane.

Then the plane in spherical is $\rho = 3/\cos\phi = 3\sec\phi$ and so we get

$$= -\int_0^{2\pi} \int_0^{\pi/6} \int_0^{3\sec\phi} (2\rho\cos\phi + 1)\rho^2 \sin\phi\ d\rho d\phi d\theta$$