Math 241 Spring 2014 Final Exam Solutions

1. (a) Find the symmetric equation of the line containing $(1,2,3)$ and $(-1,5,3)$. [10 pts]

Solution:
$\mathbf{L} = -2\,\mathbf{i} + 3\,\mathbf{j} + 0\,\mathbf{k}$ so the symmetric equation is:

$$\frac{x-1}{-2} = \frac{y-2}{3}\ ,\ z = 3$$

Note: Other common answers may have a negated $\mathbf{L}$ and may use the other point so please watch out for those!

(b) Find the distance between $(3,-5,2)$ and the plane $2x - y + 3z = 6$. Simplify. [10 pts]

Solution:
Normal vector for plane: $\mathbf{N} = 2\,\mathbf{i} - 1\,\mathbf{j} + 3\,\mathbf{k}$
Point on plane: $P = (3,0,0)$
Point off plane: $Q = (3,-5,2)$
We have

$$\overline{PQ} = 0\,\mathbf{i} - 5\,\mathbf{j} + 2\,\mathbf{k}$$

and hence

$$d = \frac{|\overline{PQ} \cdot \mathbf{N}|}{||\mathbf{N}||} = \frac{|0+5+6|}{\sqrt{4+1+9}} = \frac{11}{\sqrt{14}}$$

Note: You'll almost certainly see a variety of P and hence $\overline{PQ}$ vectors. Probably you won't see a different $\mathbf{N}$.

2. (a) For $\mathbf{u} = 2\,\mathbf{i} - \,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v} = 4\,\mathbf{i} + \,\mathbf{j} - 2\,\mathbf{k}$, find $Pr_{\mathbf{u}}\mathbf{v}$. [10 pts]

Solution:

$$\begin{aligned} Pr_{\mathbf{u}}\mathbf{v} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \\ &= \frac{8-1-6}{4+1+9}(2\,\mathbf{i} - \,\mathbf{j} + 3\,\mathbf{k}) \\ &= \frac{1}{14}(2\,\mathbf{i} - \,\mathbf{j} + 3\,\mathbf{k}) \end{aligned}$$

(b) Find the curvature $\kappa(1)$ of $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}$. [10 pts]

Solution:
We have:

$$\begin{aligned} &\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} \\ &\mathbf{v}(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} \text{ and so } \mathbf{v}(1) = 2\,\mathbf{i} + 3\,\mathbf{j} \\ &\mathbf{a}(t) = 2\,\mathbf{i} + 6t\,\mathbf{j} \text{ and so } \mathbf{a}(1) = 2\,\mathbf{i} + 6\,\mathbf{j} \end{aligned}$$

and so

$$\begin{aligned} \kappa(1) &= \frac{||\mathbf{v}(1)\times\mathbf{a}(1)||}{||\mathbf{v}(1)||^3} \\ &= \frac{||6\,\mathbf{k}||}{\left(\sqrt{4+9}\right)^3} \\ &= \frac{6}{(13)^{3/2}} \end{aligned}$$

3. (a) Find $\mathbf{T}(1)$ for $\mathbf{r}(t) = t\,\mathbf{i} - 2t^3\,\mathbf{j} + \frac{1}{t}\,\mathbf{k}$. [5 pts]

Solution:
We have

$$\mathbf{r}'(t) = 1\,\mathbf{i} - 6t^2\,\mathbf{j} - \frac{1}{t^2}\,\mathbf{k}$$

and so

$$\mathbf{T}(1) = \frac{1\,\mathbf{i} - 6\,\mathbf{j} - 1\,\mathbf{k}}{||1\,\mathbf{i} - 6\,\mathbf{j} - 1\,\mathbf{k}||} = \frac{1\,\mathbf{i} - 6\,\mathbf{j} - 1\,\mathbf{k}}{\sqrt{1+36+1}}$$

(b) Find the tangential component of acceleration for $\mathbf{r}(t) = t^3\,\mathbf{i} - 4t\,\mathbf{j} + t^2\,\mathbf{k}$ at $t = 2$. [5 pts]

Solution:
We have

$$\begin{aligned}
\mathbf{r}(t) &= t^3\,\mathbf{i} - 4t\,\mathbf{j} + t^2\,\mathbf{k} \\
\mathbf{v}(t) &= 3t^2\,\mathbf{i} - 4\,\mathbf{j} + 2t\,\mathbf{k} \text{ and so } \mathbf{v}(2) = 12\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k} \\
\mathbf{a}(t) &= 6t\,\mathbf{j} + 0\,\mathbf{j} + 2\,\mathbf{k} \text{ and so } \mathbf{a}(2) = 12\,\mathbf{i} + 0\,\mathbf{j} + 2\,\mathbf{k}
\end{aligned}$$

and so

$$\begin{aligned}
\mathbf{a}_T(2) &= \frac{\mathbf{v}(2)\cdot\mathbf{a}(2)}{||\mathbf{v}(2)||} \\
&= \frac{144+0+8}{\sqrt{144+16+16}}
\end{aligned}$$

(c) Find the point at which the line $\mathbf{r}(t) = (t+1)\,\mathbf{i} - 2t\,\mathbf{j} + (3t-2)\,\mathbf{k}$ passes through the plane $x + y - z = 10$. [10 pts]

Solution:
The line hits the point when:

$$\begin{aligned}
(t+1) + (-2t) - (3t-2) &= 10 \\
-4t &= 7 \\
t &= -7/4
\end{aligned}$$

and this is at

$$\mathbf{r}(-7/4) = -\frac{3}{4}\,\mathbf{i} + \frac{7}{2}\,\mathbf{j} - \frac{29}{4}\,\mathbf{k}$$

Hence

$$\left(-\frac{3}{4}, \frac{7}{2}, -\frac{29}{4}\right)$$

4. Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = xy$ on the circle $(x-2)^2 + y^2 = 4$. [20 pts]

Solution:

Our system of equations is

$$\begin{aligned} y &= \lambda 2(x-2) \\ x &= \lambda 2y \\ (x-2)^2 + y^2 &= 4 \end{aligned}$$

To solve the first for λ we must divide by $x - 2$. If $x - 2 = 0$ then the first says $y = 0$ and $(2, 0)$ does not satisfy the third. Thus $x - 2 \neq 0$ and so the first yields $\lambda = \frac{y}{2(x-2)}$.

Plugging this into the second yields $x = \frac{y}{2(x-2)}(2y) = \frac{y^2}{x-2}$ and so $y^2 = x(x-2)$.

Plugging this into the third yields $(x-2)^2 + x(x-2) = 4$ or $2x^2 - 6x = 0$ or $2x(x-3) = 0$ or $x = 0, 3$.

Thus the points are $(0, 0)$ and $(3, \pm\sqrt{3})$.

Then

$$\begin{aligned} f(0,0) &= 0 \\ f(3,\sqrt{3}) &= 3\sqrt{3} \qquad \text{The Maximum} \\ f(3,-\sqrt{3}) &= -3\sqrt{3} \qquad \text{The Minimum} \end{aligned}$$

5. Find and categorize all relative extrema for the function $f(x,y) = x^3 - 2xy + y^2$. [20 pts]

Solution:

We have

$$\begin{aligned} f_x &= 3x^2 - 2y = 0 \\ f_y &= -2x + 2y = 0 \end{aligned}$$

The second yields $y = x$ and so the first becomes $x(3x-2) = 0$ and hence $x = 0$ or $x = \frac{2}{3}$.

Thus the critical points are $(0,0)$ and $\left(\frac{2}{3}, \frac{2}{3}\right)$.

Then we have $D(x,y) = (6x)(2) - (-2)^2 = 12x - 4$ and so:

$D(0,0) = -$ so $(0,0)$ is a saddle point.
$D(2/3, 2/3) = +$ and $f_y(2/3, 2/3) = +$ so $(2/3, 2/3)$ is a relative minimum.

Please put problem 6 on answer sheet 6

6. Let $f(x,y) = \ln(x^2 + xy + y^2)$.

   (a) Find the direction of maximum increase of f at $(1,0)$ as a unit vector. [7 pts]

   Solution:
   We have

$$\begin{aligned}\nabla f(x,y) &= \frac{2x+y}{x^2+xy+y^2}\,\mathbf{i} + \frac{x+2y}{x^2+xy+y^2}\,\mathbf{j}\\ \nabla f(1,0) &= \frac{2}{1}\,\mathbf{i} + \frac{1}{1}\,\mathbf{j}\\ \nabla f(1,0) &= 2\,\mathbf{i} + 1\,\mathbf{j}\end{aligned}$$

   and so the unit direction is:

$$\frac{2\,\mathbf{i} + 1\,\mathbf{j}}{\sqrt{5}}$$

   (b) Find the maximum directional derivative at $(1,0)$. [6 pts]

   Solution:
   We have

$$||\nabla f(1,0)|| = \sqrt{5}$$

   (c) Calculate the directional derivative of f at $(0,1)$ in the direction of $2\,\mathbf{i} + 3\,\mathbf{j}$. [7 pts]

   Solution:
   The appropriate unit vector is $\frac{2\,\mathbf{i}+3\,\mathbf{j}}{\sqrt{13}}$ and $\nabla f(0,1) = 1\,\mathbf{i} + 2\,\mathbf{j}$ and so

$$D_{\mathbf{u}}f(0,1) = \left(\frac{2}{\sqrt{13}}\,\mathbf{i} + \frac{3}{\sqrt{13}}\,\mathbf{j}\right)\cdot(1\,\mathbf{i} + 2\,\mathbf{j}) = \frac{8}{\sqrt{13}}$$

7. (a) Find a parametrization for the part of the cylinder $y^2 + z^2 = 1$ which lies between $x = -2$ and $x = 2$. [5 pts]

Solution:
Perhaps the most obvious possibility is:

$$\begin{aligned} \mathbf{r}(x,\theta) &= x\,\mathbf{i} + \cos\theta\,\mathbf{j} + \sin\theta\,\mathbf{k} \\ &-2 \le x \le 2 \\ &0 \le \theta \le 2\pi \end{aligned}$$

(b) Find the equation of the plane tangent to the cylinder in part (a) at the point $\left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Write your answer in the form $ax + by + cz = d$. [15 pts]

Solution:
The cylinder is the level surface for $f(x,y,z) = y^2 + z^2 = 1$ and hence the normal vector is:

$$\begin{aligned} \nabla f(x,y,z) &= 0\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} \\ \nabla f\left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= 0\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + \sqrt{2}\,\mathbf{k} \end{aligned}$$

and so the plane is

$$\begin{aligned} 0(x-1) + \sqrt{2}\left(y - \frac{1}{\sqrt{2}}\right) + \sqrt{2}\left(z - \frac{1}{\sqrt{2}}\right) &= 0 \\ \sqrt{2}y + \sqrt{2}z &= 2 \end{aligned}$$

Note: A student may write $y + z = \sqrt{2}$ instead, or other variations.

8. Find the volume of the solid region D that is bounded on the sides by the upper nappe of the cone $z^2 = \frac{1}{3}(x^2 + y^2)$, on the top by the sphere $x^2 + y^2 + z^2 = 9$ and below by the sphere $x^2 + y^2 + z^2 = 1$. [20 pts]

Solution:

We have:

$$
\begin{aligned}
Volume &= \iiint_D 1 \; dV \\
&= \int_0^{2\pi} \int_0^{\pi/3} \int_1^3 \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3}\rho^3 \sin\phi \Big|_1^3 \; d\phi \; d\theta \\
&= \frac{26}{3} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi \quad d\phi \; d\theta \\
&= \frac{26}{3} \int_0^{2\pi} -\cos\phi \Big|_0^{\pi/3} \quad d\theta \\
&= \frac{26}{3} \int_0^{2\pi} \frac{1}{2} \; d\theta \\
&= \frac{13}{3} \int_0^{2\pi} 1 \; d\theta \\
&= \frac{26\pi}{3}
\end{aligned}
$$

9. Let C be the intersection curve of the parabolic sheet $y = x^2$ with the cylinder $x^2 + z^2 = 4$, oriented clockwise when viewed from the positive y-axis. Apply Stokes' Theorem to the integral $\int_C 2y\ dx + xz\ dy + z^2\ dz$ and continue until you have an iterated double integral. Do not evaluate. [20 pts]

Solution:

Stokes' Theorem gives us:

$$\int_C 2y\ dx + zy\ dy + z^2\ dz = \iint_\Sigma (-x\,\mathbf{i} + 0\,\mathbf{j} + (z-2)\,\mathbf{k}) \cdot \mathbf{n}\ dS$$

Where Σ is the portion of the parabolic sheet inside the cylinder, oriented to the left.

We parametrize Σ as

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r^2\cos^2\theta\,\mathbf{j} + r\sin\theta\,\mathbf{k}$$
$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$

And so

$$\begin{aligned}
\mathbf{r}_r &= \cos\theta\,\mathbf{i} + 2r\cos^2\theta\,\mathbf{j} + \sin\theta\,\mathbf{k} \\
\mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} - 2r^2\sin\theta\cos\theta\,\mathbf{j} + r\cos\theta\,\mathbf{k} \\
\mathbf{r}_r \times \mathbf{r}_\theta &= 2r^2\cos\theta\,\mathbf{i} - r\,\mathbf{j} + 0\,\mathbf{k}
\end{aligned}$$

This matches the orientation of Σ and hence

$$\begin{aligned}
&\iint_\Sigma (-x\,\mathbf{i} + 0\,\mathbf{j} + (z-2)\,\mathbf{k}) \cdot \mathbf{n}\ dS \\
&= + \iint_R (-r\cos\theta\,\mathbf{i} + 0\,\mathbf{j} + (r\sin\theta - 2)\,\mathbf{k}) \cdot (2r^2\cos\theta\,\mathbf{i} - r\,\mathbf{j} + 0\,\mathbf{k})\ dA \\
&= \int_0^{2\pi} \int_0^2 (-r\cos\theta)(2r^2\cos\theta) + (0)(-r) + (r\sin\theta - 2)(0)\ dr\ d\theta
\end{aligned}$$

10. (a) Evaluate $\int_C 7y\ dx + 12y\ dy$ where C is the semicircle $y = \sqrt{9-x^2}$ along with the line segment joining $(-3,0)$ with $(3,0)$, oriented clockwise. [8 pts]

Solution:

We apply Green's Theorem with a negative sign due to the orientation:

$$\begin{aligned}\int_C 7y\ dx + 12y\ dy &= -\iint_R 0 - 7\ dA \\ &= 7(\text{Area of R}) \\ &= 7\left(\frac{1}{2}\pi 3^2\right)\end{aligned}$$

(b) Find the surface area of the portion of the sphere $x^2+y^2+z^2 = 4$ inside the cylinder $x^2+y^2-2y = 0$ as an iterated double integral in r and θ. Do not evaluate. [12 pts]

Solution:

The cylinder is $x^2+(y-2)^2 = 4$ or $r = 2\sin\theta$. We therefore parametrize the top part of the surface (which we'll double) as

$$\begin{gathered}\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + \sqrt{4-r^2}\,\mathbf{k} \\ 0 \le \theta \le \pi \\ 0 \le r \le 2\sin\theta\end{gathered}$$

and so

$$\begin{aligned}\mathbf{r}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + 2r(4-r^2)^{-1/2}\,\mathbf{k} \\ \mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j} + 0\,\mathbf{k} \\ \mathbf{r}_r \times \mathbf{r}_\theta &= 2r^2\cos\theta(4-r^2)^{-1/2}\,\mathbf{i} + 2r^2\sin\theta(4-r^2)^{-1/2}\,\mathbf{j} + r\,\mathbf{k} \\ ||\mathbf{r}_r \times \mathbf{r}_\theta|| &= \sqrt{4r^4\cos^2\theta(4-r^2)^{-1} + 4r^4\sin^2\theta(4-r^2)^{-1} + r^2} \\ &= \sqrt{4r^2(4-r^2)^{-1} + r^2}\end{aligned}$$

And so

$$\begin{aligned}\text{SA} &= 2\iint_\Sigma 1\ dS \\ &= 2\iint_R \sqrt{4r^2(4-r^2)^{-1}+r^2}\ dA \\ &= 2\int_0^\pi \int_0^{2\sin\theta} \sqrt{4r^2(4-r^2)^{-1}+r^2}\ dr\ d\theta\end{aligned}$$

Note: This could be done with the shortcut formula from the book and then converted to polar. This would probably end up with the r^2 pulled to the outside as r since it arises as the Jacobian.