Math 241H - Section 0301 - Fall 2017 Name:________________________________
Quiz 8

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1. (a) [8pts] Consider the vector field $\mathbf{F} = (2xy + z^2)\mathbf{i} + x^2\mathbf{j} + (2xz + z)\mathbf{k}$. Show that $\mathbf{F}$ is conservative and find the potential function $f(x, y, z)$.

Solution: To show that $\mathbf{F}$ is conservative we write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and check

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y}.$$

It follows that $\mathbf{F}$ is conservative since $\nabla \times \mathbf{F} = \mathbf{0}$ on the whole space. To find the potential function, we need to find an $f(x, y, z)$ that satisfies

$$f_x = 2xy + z^2, \quad f_y = x^2, \quad f_z = 2xz + z.$$

To find such an f we integrate both sides of the first equation with respect to x giving,

$$f(x, y, z) = x^2y + xz^2 + h(y, z),$$

where $h(y, z)$ is some function depending on (y, x) only. Substituting this into the second equation for f gives

$$x^2 + h_y = x^2 \quad \Rightarrow \quad h_y = 0.$$

Therefore $h(y, z) = h(z)$ (h only depends on z). Finally substituting this into the third equation for f gives

$$2xz + h_z = 2xz + z \quad \Rightarrow \quad h_z = z \quad \Rightarrow \quad h = \frac{1}{2}z^2 + c,$$

where c is any constant. It follows that

$$f(x, y, z) = x^2y + xz^2 + \frac{1}{2}z^2 + c.$$

(b) [2pts] Using the same $\mathbf{F}$ as in part (a), evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where C is the positively oriented closed curve $x^2 + y^2 = 1$. (*Hint: Use the fact that* $\mathbf{F}$ *is conservative*).

Solution: Since the vector field is conservative and the line integral is around a closed contour C, we can apply the fundamental theorem of line integrals to conclude that

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0.$$