Math 241H - Section 0301 - Fall 2017 Name:____________________________
Quiz 9

This take-home quiz is to be done with out any assistance or collaboration of any kind. The assignment is due in class on Tuesday 11/28.

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1. **[4pts]** Let C be a triangle with vertices given by $(0,0)$, $(2,4)$, $(2,6)$ oriented in the counter-clockwise direction. Use Green's Theorem to compute the line integral

$$\int_C x^2 \mathrm{d}x + x\mathrm{d}y.$$

Solution: Applying Green's Theorem gives

$$\int_C x^2 \mathrm{d}x + x\mathrm{d}y = \iint_R 1\mathrm{d}A,$$

where R is the region inside the triangle. The area of this triangle is given by $\frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$. Therefore

$$\int_C x^2 \mathrm{d}x + x\mathrm{d}y = 2.$$

2. **[3pts]** Let $f(x,y)$ be a smooth function, $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ be a smooth vector field and R be a simple region in the x,y plane bounded by the closed curve C oriented in the positive (counter-clockwise) direction with outward facing unit normal $\mathbf{n}_C(x,y)$. Use Green's Theorem to show the following generalized integration by parts formula

$$\iint_R \nabla f \cdot \mathbf{F}\, \mathrm{d}A = -\iint_R f\, \nabla \cdot \mathbf{F}\, \mathrm{d}A + \int_C f\, \mathbf{F} \cdot \mathbf{n}_C\, \mathrm{d}s,$$

Hint: Show that

$$\int_C f\, \mathbf{F} \cdot \mathbf{n}_C\, \mathrm{d}s = \int_C -f\, N\mathrm{d}x + f\, M\mathrm{d}y$$

and apply Green's Theorem to the line integral on the right-hand side.

Solution: Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ be a parameterization of C with $a \le t \le b$. The outward facing normal $\mathbf{n}_C$ can be parameterized by

$$\mathbf{n}_C = y'(t)\mathbf{i} - x'(t)\mathbf{j}.$$

Therefore the line integral is given by

$$\begin{aligned}\int_C f\, \mathbf{F} \cdot \mathbf{n}_C\, \mathrm{d}s &= \int_a^b f(x(t),y(t))\, M(x(t),y(t))y'(t) - f(x(t),y(t))\, N(x(t),y(t))x'(t)\mathrm{d}t \\ &= \int_C -fN\mathrm{d}x + fM\mathrm{d}y\end{aligned}$$

We now apply Green's theorem to this line integral to obtain

$$\begin{aligned}
\int_C f\,\mathbf{F}\cdot\mathbf{n}_C\,\mathrm{d}s &= \int_C -fN\mathrm{d}x + fM\mathrm{d}y \\
&= \iint_R \frac{\partial}{\partial y}(fM) + \frac{\partial}{\partial x}(fN)\mathrm{d}A \\
&= \iint_R f_y N + fN_y + f_x M + fM_x \mathrm{d}A \\
&= \iint_R f(M_x + N_y)\mathrm{d}A + \iint_R f_x M + f_y N\mathrm{d}A \\
&= \iint_R f\nabla\cdot\mathbf{F}\mathrm{d}A + \iint_R \nabla f\cdot\mathbf{F}\mathrm{d}A.
\end{aligned}$$

We have therefore derived the desired formula.

3. **[3pts]** Let $\mathbf{G}(x,y) = P(x,y)\mathbf{k}$ and $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ be smooth vector fields and let R be a simple region in the x,y plane bounded by the closed curve C oriented in the positive (counter-clockwise) direction. Denote the unit normal to R induced by the orientation of C by $\mathbf{n}_R = \mathbf{k}$. Use Green's Theorem to show another generalized integration by parts formula

$$\iint_R \mathbf{G}\cdot(\nabla\times\mathbf{F})\mathrm{d}A = \iint_R (\nabla\times\mathbf{G})\cdot\mathbf{F}\,\mathrm{d}A + \int_C (\mathbf{G}\cdot\mathbf{n}_R)\,\mathbf{F}\cdot\mathrm{d}\mathbf{r}.$$

Hint: Show that

$$\int_C (\mathbf{G}\cdot\mathbf{n}_R)\,\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_C P\,M\mathrm{d}x + P\,N\mathrm{d}y,$$

and apply Green's Theorem to the line integral on the right-hand side. Write out $\nabla\times\mathbf{G}$ and $\nabla\times\mathbf{F}$ explicitly clarify the expresions involving the curl.

Solution: Since $\mathbf{n}_R = \mathbf{k}$ we see that

$$(\mathbf{G}\cdot\mathbf{n}_R)\mathbf{F} = P\mathbf{F} = PM\mathbf{i} + PN\mathbf{j}.$$

Therefore we may apply Green's theorem to the line integral

$$\begin{aligned}
\int_C (\mathbf{G}\cdot\mathbf{n}_R)\mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_C PM\mathrm{d}x + PN\mathrm{d}y \\
&= \iint_R \frac{\partial}{\partial x}(PN) - \frac{\partial}{\partial y}(PM)\mathrm{d}A \\
&= \iint_R (P_x N - P_y M)\mathrm{d}A + \iint_R P(N_x - M_y)\mathrm{d}A.
\end{aligned}$$

Since

$$\nabla\times\mathbf{G} = P_y\mathbf{i} - P_x\mathbf{j},$$

and

$$\nabla \times \mathbf{F} = (N_x - M_y)\mathbf{k},$$

we can write

$$\iint_R (P_x N - P_y M) \mathrm{d}A = -\iint_R (\nabla \times \mathbf{G}) \cdot \mathbf{F} \mathrm{d}A$$

and

$$\iint_R P(N_x - M_y) \mathrm{d}A = \iint_R \mathbf{G} \cdot (\nabla \times \mathbf{F}) \mathrm{d}A.$$

Putting everything together gives

$$\int_C (\mathbf{G} \cdot \mathbf{n}_R) \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\iint_R (\nabla \times \mathbf{G}) \cdot \mathbf{F} \mathrm{d}A + \iint_R \mathbf{G} \cdot (\nabla \times \mathbf{F}) \mathrm{d}A,$$

which is the equation to be proved.